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# RAREFIED GAS MOTION IN A SHORT PLANAR CHANNEL OVER THE ENTIRE KNUDSEN NUMBER RANGE

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It was demonstrated in [1] that flow of a rarefied gas in a finite channel has been considered only over a narrow Knudsen number range or in coarse approximations valid only for sufficiently long channels. In that study the problem was solved for a wide range, but in the approximation that molecules entering the channel through its faces have an absolutely Maxwellian distribution function, which also limits application of its results to finite, although sufficiently long channels. In connection with this there is a need for a precise solution of the given problem over the entire range of Knudsen numbers with consideration of flow formation in the region of the vessel near the input.

1. We will consider a planar channel of length $\ell$, height $2a$, infinite in the z-direction, connecting two semi-infinite vessels of one and the same gas (Fig. 1). Within the vessels at a sufficient distance from the channel the gas is maintained under equilibrium conditions at pressures $p_1$ and $p_2$ and identical temperatures T. Under the action of the pressure head the gas moves in the x-direction.

We introduce the scale factors: $a$, $n_1$, $\beta^{1/2} = (2RT)^{1/2}$, $n_1\beta^{-3/2}$, $\eta_1 = n_1 m v \lambda_1/2$ for the length, density n, velocities $\mathbf{c}$ and $\mathbf{u}$, distribution function f, and viscosity coefficient $\eta$. Here R is the ideal gas constant, m is the mass of a molecule, $v = (8RT/\pi)^{1/2}$ is the thermal velocity of a molecule, $\lambda_1$ is the molecular free path length in the first vessel. All further expressions will be written using these scaling factors.

We assume that the relative pressure head is much less than unity ($|p_2 - p_1|/p_1 \ll 1$) and that all gas molecules are reflected from the walls of the channel and vessels diffusely. For the distribution function equation we use the BGK model of the Boltzmann equation [2]

$$\mathbf{c}\partial f/\partial \mathbf{r} = \delta(f^0 - f),$$

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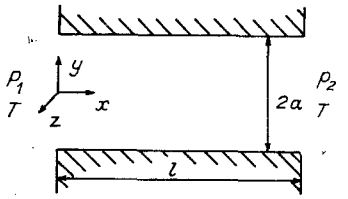


Fig. 1

where $\delta = \sqrt{\pi}a/2\lambda_1$ is the reciprocal Knudsen number, $f^0(\mathbf{r}, \mathbf{c}) = (n(\mathbf{r})/\pi^{3/2}) \exp[-(\mathbf{c} - \mathbf{u}(\mathbf{r}))^2]$; $n(\mathbf{r}) = \int\int\int_{-\infty}^{\infty} f(\mathbf{r}, \mathbf{c})\, d\mathbf{c}$; $\mathbf{u}(\mathbf{r}) = \frac{1}{n}\int\int\int_{-\infty}^{\infty} f(\mathbf{r}, \mathbf{c})\, \mathbf{c}\, d\mathbf{c}$; $\mathbf{c}$ is the velocity of a molecule, $\mathbf{r} = \mathbf{r}(x, y)$.

A procedure was presented in [1] for linearizing the BGK equation and deriving integral equations for moments of the distribution function. Therefore we will present only the final result:

$$q_i(\mathbf{r}) = \int\int \sum_{j=1}^{3} K_{ij}(\mathbf{r}, \mathbf{r}')\, q_j(\mathbf{r}')\, d\mathbf{r}' + \int K_{i4}(\mathbf{r}, x')\, q_4(x')\, dx' + \Phi_i(\mathbf{r})$$

$$(i = 1, 2, 3, 4),$$

$$q_1 = (n - 1)p_1/\Delta p,\ q_2 = u_x p_1/\Delta p,\ q_3 = u_y p_1/\Delta p,\ q_4(x) = (n_W(x) - 1)p_1/\Delta p$$

($n_W$ is the reduced number of molecules incident upon a unit area per unit time). We will also omit the cumbersome expressions for $K_{ij}$ and $\Phi_i$, since they are also presented in [1]. The given integral equations differ from those of [1] in that here we have introduced integration over the portion "visible" from the point (x, y) of the region including both the cavity channel and the vessel. The integral equations obtained can be solved by the Krylov–Bogolyubov method [1]. But the presence of infinite pre-entrance regions creates significant computation difficulties. This is related first, to an increase in the number of cells into which the flow field is divided. Second, in the numerical calculation it was necessary to limit ourselves to a finite pre-entrance region, one such that required calculation accuracy for the flow field would be assured at least within the channel and in regions adjacent to the channel input sections. As analysis of the results obtained will show, for an accuracy of 2% it is necessary that the boundary in the pre-entrance region be removed from the channel input section by not less than 40 molecular free path lengths and not less than a distance $4a$.

With increase in channel length $L = l/a$ and rarefaction parameter $\delta$ the numerical difficulties increase severely, so that special study of the corresponding limiting cases is necessary.

2. When the channel length is much greater than the molecular free path length ($\delta L \gg 1$), the flow fields near the faces and in the pre-entrance regions become similar at fixed $\delta$ and different L. As was shown in [1], this can be used as a basis to obtain a relationship between the flow fields and flow rates in channels of different length, thus avoiding computer calculation of a large number of variants, which would require significant machine time expenditures. We write the reduced gas flow rate in the form

$$G = \frac{p_1 L}{\Delta p}\int_{-1}^{1} u_x(x, y)\, dy = \frac{L}{L + \Delta L} G_\infty \qquad (2.1)$$

(where $G_\infty$ is the reduced flow rate in an infinite channel, and $\Delta L$ is a value independent of channel length which is definable from analysis of the flow field [1]). It is evident from analysis of numerical calculations that for an accuracy of 2% the necessary condition for use of Eq. (2.1) is the inequality $\delta L \geq 20$, $L \geq 4$. Table 1 presents values of $G_\infty$ and $\Delta L$ for certain $\delta$. We will note that the minimum channel length for which the above considerations are valid may be comparable to $\Delta L$. It is evident from Table 1 that the divergence from flow rate calculation results obtained directly for an infinite channel [3] does not exceed the calculation uncertainty.

3. In the limiting case of a continuous medium ($\delta \to \infty$) the gas flow is described by the system of Navier–Stokes equations [4], to which the continuity equation is added. In solving the kinetic equation we have limited ourselves to the approximation linear in pres-

TABLE 1

| $\delta$ | $L_{min}$ | $\Delta L$ | $G_\infty$ | $G_\infty$ [3] |
|---|---|---|---|---|
| 0,1 | 200 | 7,04 | 1,79 | 1,8079 |
| 0,2 | 100 | 5,34 | 1,63 | 1,6408 |
| 0,5 | 40 | 4,04 | 1,53 | 1,5389 |
| 1,0 | 20 | 3.26 | 1,58 | 1,5942 |
| 2,0 | 10 | 2,86 | 1,84 | 1,8440 |
| 4,0 | 5 | 2,88 | 2,43 | 2,4472 |

sure head. Therefore in the equations of the mechanics of a continuous medium we retain only terms of first order smallness in $\Delta p/p_1$. Using the dimensionless quantities, we obtain

$$(1/\delta)(\partial^2 u_x/\partial x^2 + \partial^2 u_x/\partial y^2) = (1/p_1)\partial p/\partial x,$$
$$(1/\delta)(\partial^2 u_y/\partial x^2 + \partial^2 u_y/\partial y^2) = (1/p_1)\partial p/\partial y,\ \partial u_x/\partial x + \partial u_y/\partial y = 0. \tag{3.1}$$

On the channel and vessel walls we apply the attachment condition $\mathbf{u} = 0$. At infinite distance from the channel in both vessels the gas has different pressures $p_1$ and $p_2$.

To solve system (3.1) it will be convenient to transform to the variables of vorticity $\omega$ and flow function $\psi$ [5]:

$$\omega = (1/\delta)(\partial u_x/\partial y - \partial u_y/\partial x),\ (1/\delta)u_x = \partial\psi/\partial y,\ (1/\delta)u_y = -\partial\psi/\partial x.$$

It can easily be shown that system (3.1) corresponds to the system of [5]

$$\Delta\omega = 0,\quad \Delta\psi = \omega. \tag{3.2}$$

Since the flow field is symmetric [1], system (3.2) was solved in the upper right quarter of the channel and vessel. In this case the boundary conditions take on the form

$$\begin{aligned}
&x = L/2, && 0 \leqslant y \leqslant 1, && \partial\psi/\partial x = 0, && \partial\omega/\partial x = 0;\\
&L/2 \leqslant x \leqslant \infty, && y = 0, && \psi = 1/2, && \omega = 0;\\
&L/2 \leqslant x \leqslant L, && y = 1, && \psi = 0, && \partial\psi/\partial y = 0;\\
&x = L, && 1 \leqslant y \leqslant 0, && \psi = 0, && \partial\psi/\partial x = 0;\\
&\sqrt{x^2 + y^2} \to \infty, && \partial\psi/\partial x = 0, && && \partial\psi/\partial y = 0.
\end{aligned} \tag{3.3}$$

The function $\psi(x, y)$ is defined to the accuracy of an arbitrary constant, so that along the boundary of the flow field this constant is taken equal to zero. In light of the attachment condition the normal derivative of $\psi$ must equal zero on the entire gas flow boundary. On the axis of symmetry $y = 0$, $\psi = 1/2$. This implies that the gas flow rate through the channel cross section is always equal to unity. The unknown quantity, the pressure head

$$\frac{\Delta p}{p_1} = \frac{2}{\delta}\int_{L/2}^{\infty} \frac{\partial\omega}{\partial y}\Big|_{y=0} dx. \tag{3.4}$$

Knowing the pressure head we find G, proportional to $\delta$.

In the numerical solution of Eq. (3.2), as in the case of the kinetic equation, it is necessary to limit oneself to a finite flow region. The dimensions of the region considered can be decreased if in place of the last condition of Eq. (3.3) upon the boundary we use the asymptote of the flow field, which can be obtained by taking the input section as a point mass source. Then for $\omega$ and $\psi$ we have

$$\omega = \frac{4y(x-L)}{\pi[(x-L)^2 - y^2]},\quad \psi = -\frac{1}{\pi}\left\{\frac{(x-L)y}{(x-L)^2 + y^2} + \operatorname{arctg}\frac{y}{x-L}\right\} + \frac{1}{2}.$$

Calculations show that for 2% accuracy the dimensions of the flow region considered must be five times the height of the channel.

System (3.2) with boundary conditions (3.3) was solved on a regular grid with a difference scheme

$$\omega_{ij} = \frac{1}{4}(\omega_{i-1,j} + \omega_{i+1,j} + \omega_{i,j-1} + \omega_{i,j+1}),$$

$$\psi_{ij} = \frac{1}{4}(\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1} - \omega_{ij}\Delta x^2)$$

$$(\omega_{ij} = \omega(x_i, y_j),\ \psi_{ij} = \psi(x_i, y_j),\ \Delta x = x_{i+1} - x_i).$$

TABLE 2

| L | A | σ |
|---|---|---|
| 2 | 0,1816 | 0,3088 |
| 10 | 0,2854 | 0,8444 |
| 20 | 0,3076 | 1,016 |
| 60 | 0,3244 | 1,155 |
| ∞ | 0,3334 | 1,257 |

TABLE 3

| δ | G | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | L | | | | | | | |
| | 2 | | 10 | | 20 | | 60 | |
| 0,01 | 0,391 | 1 % | 1,00 | 0 % | 1,35 | 1 % | 1,88 | 0 % |
| 0,1 | 0,421 | 9 % | 1,02 | 5 % | 1,28 | 3 % | 1,57 | 1 % |
| 1 | 0,601 | 36 % | 1,19 | 19 % | 1,36 | 12 % | 1,50 | 5 % |
| 4 | 1,00 | 60 % | 1,89 | 37 % | 2,13 | 25 % | 2,32 | 11 % |

Knowing the dependence of ω on coordinates and using Eq. (3.4), we have $G = Lp_1/\Delta p = A\delta$.

Table 2 presents values of the proportionality coefficient A. For $L = \infty$, this quantity can be obtained analytically, by direct solution of Eq. (3.1).

To merge the results of solution of the kinetic equation and the Navier–Stokes equation we represent the gas flow rate as

$$G = \delta(A + \sigma\lambda/a). \tag{3.5}$$

The quantity σ characterizes the contribution of slip to the flow rate. We find its value from the condition that the flow rate obtained by solution of the kinetic equation at maximum $\delta = 4$ has the form of Eq. (3.5). It is evident from Table 2 that with increase in channel length the contribution of slip to the flow rate increases. Estimates show that the uncertainty of Eq. (3.5) does not exceed 2.5% in the range $\delta \geq 4$.

4. Results of flow rate calculation both directly by solution of the kinetic equation, and by Eqs. (2.1) and (3.5) are shown in Fig. 2. In the free molecular regime the flow rate coincides with that of [6] within the limits of its accuracy. For comparison the dashed line shows the results of [1], which did not consider a change in the distribution function in the pre-entrance region. For various values of L and δ, Table 3 shows exact flow rate

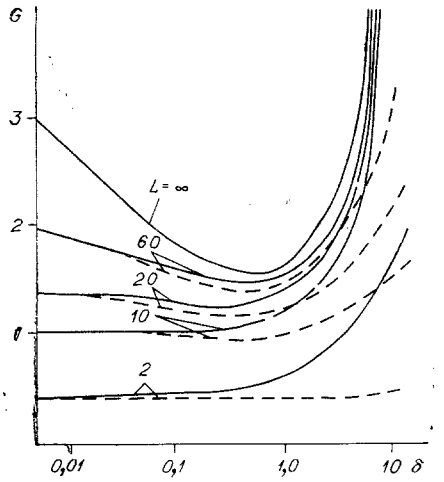

Fig. 2

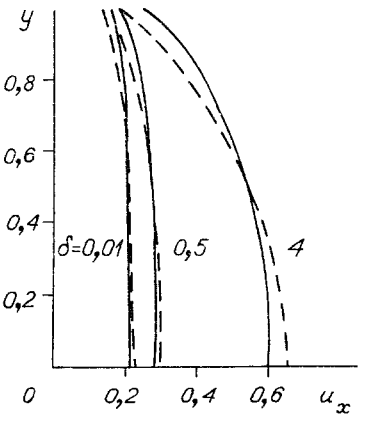

Fig. 3

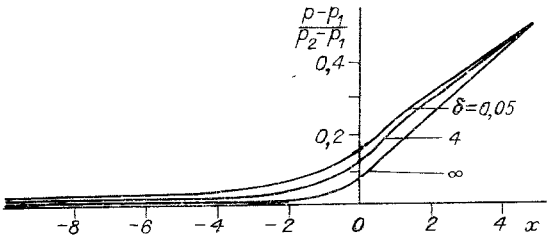

Fig. 4

values (left-hand column for each L) and relative difference between precise and approximate values from [1] (right-hand columns). It is evident from Fig. 2 and Table 3 that for short channels and large $\delta$ the pre-entrance region exerts a significant effect on gas flow formation.

We will note that for large channel lengths ($L > 10$) the dependence of G upon $\delta$ is nonmonotonic. For $\delta \sim 0.5$ there is a minimum, which has been termed the Knudsen minimum; at $L \leq 10$ this minimum disappears. Of more than slight interest is the flow rate through a slit ($L = 0$). This case was considered in detail in [7]. Here we will limit ourselves to presentation of expressions which interpolate the numerical and analytical results over the entire Knudsen number range:

$$M/M_0 = 1 - 0.15\delta \ln \delta/(1 + 0.004\delta \ln \delta) + (\pi^{3/2}/8)\delta$$

($M$, $M_0$ are the gas mass flow rate at arbitrary number $\delta$ and in the free molecular regime).

5. We will consider the flow field. Figure 3 shows the change in velocity profile for various $\delta$ in a channel of length $L = 2$. In the regime close to free molecular and up to the intermediate regime the profiles are close to homogeneous. With increase in $\delta$ after the intermediate regime the degree of inhomogeneity increases severely. The relative difference in velocity profiles in various channel sections depends weakly on $\delta$. For $\delta = 0.01$ on the channel axis the velocity changes by 5%, and at $\delta = 4$, by 7%. The velocity profile behaves in an analogous manner for other channel lengths.

Pressure is shown as a function of longitudinal coordinate on the channel axis for $L = 10$ in Fig. 4, where for all $\delta$ in the channel the function is linear, the slope of the straight line depending weakly on $\delta$. In the pre-entrance region at sufficiently large distances from the channel the pressure tends toward its value at infinity by a $1/x^2$ law.

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